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SOME FIXED-POINT THEOREMS RESULT ON BANACH ALGEBRAS

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In this paper, we prove some fixed point results in triplet of self-mappings on Banach algebras and application of nonlinear functional-integral equation are introduced. In addition, we are provided some examples to demonstrate the application of our findings and we used the new simplified technique to find the result of fixed point theorem. Our results are generalized results of Das [7].

INTRODUCTION

Fixed point theory is one of the most effective and powerful tools of contemporary mathematics and might be regarded as one of the fundamental concepts in nonlinear analysis. General and algebraic topology, synthetic, analytic, metric and differential geometry as well as pure and applied analysis are all well-combined in fixed point theory. An important area of nonlinear function analysis is the study of fixed points for nonlinear mappings and nonlinear integral equations and differential equations are widely used to apply the theory of fixed points. Some issues can be solved with the help of fixed-point theorems, which offer the prerequisites for the existence of solutions for some transformations. The use of fixed-point approaches has greatly improved physics, chemistry, biology, engineering, economics and many other subjects.

In 1988, Dhage first used fixed point theorems to Banach algebras. Dhage has written many papers [3, 4, 5] that explore non-linear integral equations using fixed point theorems in Banach algebras. In 2010, Amar et al. [1], introduced a class of Banach algebras satisfying certain sequential conditions and gave applications of non-linear integral equation using fixed point theorems under certain conditions. Pathak and Deepmala [12] define P-Lipschitzian maps in 2012 and deduced various examples of Dhage's fixed-point theorem on a Banach algebra. Many scholars, including Mishra et al. [19, 20, 21, 22], Deepmala [8, 9], Mishra [29] etc., demonstrated certain results regarding the existence of solutions for some nonlinear functional integral equations in Banach algebra and some intriguing results.

In Hausdorff topological vector space, Hadzic [23] proved an extension of the Rzepecki fixed point theorem in 1982. In [24, 25, 26], Vijayaraju established the validity of the sum of two mappings in reflexive Banach spaces as well as the fixed points asymptotic 1-set contraction mapping in real Banach spaces.

In this paper, we consider a triplet of self-mappings ($M1$, $M2$, $M3$) on a Banach algebra X with a subset A and investigate the circumstances in which the operator equation $v = M1vM2vM3v$ has a solution in A. Here, with some appropriate examples, a possible application of the findings to the tensor product of Banach algebras is also presented. A nonlinear functional-integral equation application should also be proved.

PRELIMINARIES

In order to begin this work, it is necessary to quickly review the following definitions, concepts and ideas.

Def. 1 Let X be a Banach algebra in which the operation of multiplication is defined as follows: for all $x, y, z \in$ X , $\alpha \in$ R

1) $(xy)z = x(yz)$,

2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$,

3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

(4) $||xy|| \le ||x||.||y||$.

Def. 2 Let a Banach algebra X has a unit e (i.e., a multiplicative identity) such that $ey = ye = y$ for all $y \in$ X. An element $y \in$ X is said to be invertible if there is an inverse element $z \in$ X such that $yz = zy = e$. The inverse of y is denoted by $y-1$.

Def. 3 [27] Let M is demiclosed if $\{xn\} \subset A$(M), $xn \rightarrow x$ and M(xn) $\rightarrow y$ (weakly) implies $x \in A$(M) and M$x = y$.

Def. 4 [27] Let M is closed if $\{xn\} \subset A$(M), $xn \rightarrow x$ and M(xn) $\rightarrow y$ implies $x \in A$(M) and M$x = y$.

Def. 5 [28] Let M is said to be demicompact at a if for any bounded sequence $\{xn\}$ in A such that xn – M$xn \rightarrow a$ as $n \rightarrow \infty$, there exists a subsequence x_{n_i} and a point b in A such that $x_{n_i} \rightarrow b$ as $i \rightarrow \infty$ and b – M(b) = a.

Def. 6 ([16], [17]) Let M : $A \rightarrow A$ be a mapping.

(1) M is said to be uniformly L-Lipschitzian if there exist $L > 0$ such that $x, y \in A$

$$||Mnx - Mny|| \le L||x - y||, \forall\ n \in \mathbb{N}$$

(2) M is said to be asymptotically nonexpansive if there exist a sequence $bn \subset [1, \infty)$ with $bn \rightarrow 1$ such that, for any $x, y \in A$

$$||Mnx - Mny|| \le bn||x - y||, \forall\ n \in \mathbb{N}$$

Def. 7 Let X be a Banach algebra and M1, M2 be two self mappings on X. Then M1, M2 are said to satisfy the nonvacuous condition if for every sequence $\{xn\} \subset$ X the operator equation limn→∞ M1(v)M2(xn) = v, v $\in$ X has one and only one solution (xn)0 in X.

Algebric tensor product: [9] Let X,Y be normed spaces over F with dual spaces X*and Y* respectively. Given $x \in$ X, $y \in$ Y. Let $x \otimes y$ be the element of BL(X*, Y*; F) (which is the set of all bounded bilinear forms from X* × Y* to F), defined by

$$x \otimes y\ (f, g) = f(x)g(y),\ (f \in X^*, g \in Y^*)$$

The algebraic tensor product of X and Y, X $\otimes$ Y is defined to be the linear span of $\{x \otimes y :\ x \in$ X, $y \in$ Y$\}$ in BL(X*, Y*; F)

Projective tensor norm: [9] Given normed spaces X and Y, the projective tensor norm γ on X $\otimes$ Y is defined by

$\| u \|\gamma = \inf\{ \sum_i \|x_i\| . \|y_i\| : u = \sum_i x_i \otimes y_i \}$

where all finite representations of u are taken to have the infimum.

The completion of (X ⊗ Y, γ) is called projective tensor product of X and Y and it is denoted by X ⊗γ Y.

MAIN RESULTS

Theorem 1: Let *A* be a non-empty compact convex subset of a Banach Algebra X and let (M1, M2, M3) be a triplet of self-mapping on *A* such that

(a) M1, M2 and M3 are continuous,

(b) M1vM2vM3v ∈ *A* for all v ∈ *A*

Then the operator equation v = M1vM2vM3v has a solution in *A*.

Proof. We define *F*: *A* → *A* by *F*(v) = M1vM2vM3v. Let {*p*n} be a sequence in *A* converging to a point *p*.

So, *p* ∈ *A* as *A* is closed. Now,

|| *F*(v) – *F*(w) – *F*(x) || = || M1vM2vM3v – M1wM2wM3w – M1xM2xM3x ||
≤ || M1v – M1w– M1x || ||M2vM3v|| +|| M2v – M2w– M2x ||
||M1wM3w|| + || M3v – M3w – M3x || ||M1xM2x||

Since M1, M2 and M3 are continuous so, *F* is continuous. By an application of Schauder's fixed point theorem, we have fixed point for *F*. Hence the operator equation v = M1vM2vM3v has a solution in *A*.

Corollary 1: Let *AX*, *AY*, *AZ* and *AX* ⊗ *AY* ⊗ *AZ* be closed, convex and bounded subsets of Banach algebras X,
Y, Z and X ⊗γ Y ⊗γ Z respectively. Let (M1, M2, M3) be a triplet of self-mapping on *AX* ⊗ *AY* ⊗ *AZ* such that

(a) M1, M2 and M3 are completely continuous,

(b) M1vM2vM3v ∈ *AX* ⊗ *AY* ⊗ *AZ* for all v ∈ *AX* ⊗ *AY* ⊗ *AZ*

then the operator equation v = M1vM2vM3v has a solution in *AX* ⊗*AY* ⊗ *AZ*.

Example 1: Let *Ai*, *AJ*, *Ak* and *Ai* ⊗ *AJ* ⊗ *Ak* be subsets of Banach algebras i, J, k and i ⊗γ J ⊗γ k respectively.

Define

Ai = {x ∈ *Ai* : ||x||≤ K1}, *AJ* = { y ∈ *AJ* : ||y||≤ K2} and *Ak* = {z ∈ *Ak* : ||z||≤ K3}

then clearly *Ai*, *AJ*, *Ak* and *Ai* ⊗ *AJ* ⊗ *Ak* are closed, convex and bounded.

We define M1, M2, M3 : *Ai* ⊗γ *AJ* ⊗γ *Ak* → *Ai* ⊗γ *AJ* ⊗γ *Ak* by $M1\left(\sum_i a_i \otimes x_i \otimes y_i\right)$

$$= \sum_i \left\{ \frac{a_{i_n} x_i y_i}{n} \right\}_n = M2\left(\sum_i a_i \otimes x_i \otimes y_i\right)$$

$$= M3\left(\sum_i a_i \otimes x_i \otimes y_i\right), \text{ where } a_i = \{a_{i_n}\}_n . [\, i \otimes_\gamma X \otimes_\gamma Y = i(X).i(Y) \,].$$

To show that M1 is compact:

Let Ms : *Ai* ⊗γ *AJ* ⊗γ *Ak* → *Ai* ⊗γ *AJ* ⊗γ *Ak* be defined by

$$Ms\left(\sum_i a_i \otimes x_i \otimes y_i\right) = \sum_i \left\{ a_{i_1} x_i y_i, \frac{a_{i_2} x_i y_i}{2}, \frac{a_{i_3} x_i y_i}{3}, \ldots\ldots, \frac{a_{i_m} x_i y_i}{m}, 0,0,0\ldots\ldots \right\}$$

Then each Ms is linear, bounded and compact [6]. Also,

$$\|(Ms - M1)\left(\sum_i a_i \otimes x_i \otimes y_i\right)\| = \|\sum_i \left\{a_{i_1}x_iy_i, \frac{a_{i_2}x_iy_i}{2}, \frac{a_{i_3}x_iy_i}{3}, ..., \frac{a_{i_m}x_iy_i}{m}, 0,0,0......\right\}$$

$$-$$

$$\sum_i \left\{a_{i_1}x_iy_i, \frac{a_{i_2}x_iy_i}{2}, \frac{a_{i_3}x_iy_i}{3}, ..., \frac{a_{i_m}x_iy_i}{m}, \frac{a_{i_{m+1}}x_iy_i}{m+1}, ...\right\}\|$$

$$= \|\sum_i \left\{0,0,......0, \frac{a_{i_{m+1}}x_iy_i}{m+1}, \frac{a_{i_{m+2}}x_iy_i}{m+2}, ...\right\}\|$$

$$\leq \sum_i \sum_{j=m+1}^{\infty} \frac{1}{j}|a_{ij}|.|x_i|.|y_i| < \frac{1}{m+1}\sum_i \sum_{j=m+1}^{\infty}|a_{ij}|.|x_i|.|y_i|$$

$$\leq \frac{1}{m+1}\sum_i \sum_{j=1}^{\infty}|a_{ij}|.|x_i|.|y_i| = \frac{1}{m+1}\sum_i \|a_i\|.|x_i|.|y_i|$$

So, taking the projective tensor norm,

$$\|(Ms - M1)\left(\sum_i a_i \otimes x_i \otimes y_i\right)\| < \frac{1}{m+1}\|\sum_i a_i \otimes x_i \otimes y_i\|$$

Therefore, Ms → M1 and so, M1 is compact.
To show that M2 is compact:
Let Ms : $Ai \otimes\gamma AJ \otimes\gamma Ak \rightarrow Ai \otimes\gamma AJ \otimes\gamma Ak$ be defined by

$$Ms\left(\sum_i a_i \otimes x_i \otimes y_i\right) = \sum_i \left\{a_{i_1}x_iy_i, \frac{a_{i_2}x_iy_i}{2}, \frac{a_{i_3}x_iy_i}{3}, ..., \frac{a_{i_m}x_iy_i}{m}, 0,0,0......\right\}$$

Then each Ms is linear, bounded and compact [6]. Also,

$$\|(Ms - M2)\left(\sum_i a_i \otimes x_i \otimes y_i\right)\| = \|\sum_i \left\{a_{i_1}x_iy_i, \frac{a_{i_2}x_iy_i}{2}, \frac{a_{i_3}x_iy_i}{3}, ..., \frac{a_{i_m}x_iy_i}{m}, 0,0,0......\right\}$$

$$-$$

$$\sum_i \left\{a_{i_1}x_iy_i, \frac{a_{i_2}x_iy_i}{2}, \frac{a_{i_3}x_iy_i}{3}, ..., \frac{a_{i_m}x_iy_i}{m}, \frac{a_{i_{m+1}}x_iy_i}{m+1}, ...\right\}\|$$

$$= \|\sum_i \left\{0,0,......0, \frac{a_{i_{m+1}}x_iy_i}{m+1}, \frac{a_{i_{m+2}}x_iy_i}{m+2}, ...\right\}\|$$

$$\leq \sum_i \sum_{j=m+1}^{\infty} \frac{1}{j}|a_{ij}|.|x_i|.|y_i| < \frac{1}{m+1}\sum_i \sum_{j=m+1}^{\infty}|a_{ij}|.|x_i|.|y_i|$$

$$\leq \frac{1}{m+1}\sum_i \sum_{j=1}^{\infty}|a_{ij}|.|x_i|.|y_i| = \frac{1}{m+1}\sum_i \|a_i\|.|x_i|.|y_i|$$

So, taking the projective tensor norm, $\|(Ms - M2)\left(\sum_i a_i \otimes x_i \otimes y_i\right)\|$

$$< \frac{1}{m+1}\|\sum_i a_i \otimes x_i \otimes y_i\|$$

Therefore, Ms → M2 and so, M2 is compact.
To show that M3 is compact:
Let Ms : $Ai \otimes\gamma AJ \otimes\gamma Ak \rightarrow Ai \otimes\gamma AJ \otimes\gamma Ak$ be defined by

$$Ms\left(\sum_i a_i \otimes x_i \otimes y_i\right) = \sum_i \left\{a_{i_1}x_iy_i, \frac{a_{i_2}x_iy_i}{2}, \frac{a_{i_3}x_iy_i}{3}, ..., \frac{a_{i_m}x_iy_i}{m}, 0,0,0......\right\}$$

Then each Ms is linear, bounded and compact [6]. Also,

$$||(\mathrm{Ms} - \mathrm{M3})\left(\sum_i a_i \otimes x_i \otimes y_i\right)|| = ||\sum_i \left\{a_{i_1} x_i y_i, \frac{a_{i_2} x_i y_i}{2}, \frac{a_{i_3} x_i y_i}{3}, \ldots\ldots, \frac{a_{i_m} x_i y_i}{m}, 0,0,0\ldots\ldots\right\}$$

$$-$$

$$\sum_i \left\{a_{i_1} x_i y_i, \frac{a_{i_2} x_i y_i}{2}, \frac{a_{i_3} x_i y_i}{3}, \ldots\ldots, \frac{a_{i_m} x_i y_i}{m}, \frac{a_{i_{m+1}} x_i y_i}{m+1}, \ldots\ldots\right\}||$$

$$= ||\sum_i \left\{0,0,\ldots\ldots0, \frac{a_{i_{m+1}} x_i y_i}{m+1}, \frac{a_{i_{m+2}} x_i y_i}{m+2}, \ldots\ldots\right\}||$$

$$\leq \sum_i \sum_{j=m+1}^{\infty} \frac{1}{j} |a_{ij}| . |x_i| . |y_i| < \frac{1}{m+1} \sum_i \sum_{j=m+1}^{\infty} |a_{ij}| . |x_i| . |y_i|$$

$$\leq \frac{1}{m+1} \sum_i \sum_{j=1}^{\infty} |a_{ij}| . |x_i| . |y_i| = \frac{1}{m+1} \sum_i \|a_i\| . |x_i| . |y_i|$$

So, taking the projective tensor norm,

$$||(\mathrm{Ms} - \mathrm{M3})\left(\sum_i a_i \otimes x_i \otimes y_i\right)|| < \frac{1}{m+1} ||\sum_i a_i \otimes x_i \otimes y_i||$$

Therefore, Ms → M3 and so, M3 is compact.

Thus M1, M2 and M3 are compact. Since every compact operator in Banach space is completely continuous, so M1, M2 and M3 are completely continuous. Then, by **Corollary 1,** the operator equation has a solution.

Theorem 2: Let X be non-empty Banach Algebra and let M1, M2, M3 be three self-mapping on X such that

(a) H is homomorphism and it has a unique fixed point

(b) M1H = HM1, M2H = HM2 and M3H = HM3

then the unique fixed point of H is a solution of the operator equation v = M1vM2vM3v in X.

Proof. We define *F*: X → X by *F*(v) = M1vM2vM3v. Let p be the unique fixed point on H.

Now, *F*(H(v)) = M1(H(v))M2(H(v))M3(H(v)) = H(M1(v))H(M2(v))H(M3(v)) = H(M1vM2vM3v) = H(*F*v)

Hence, H(*F*p) = *F*(H(p)) = *F*p so *F*p = p as H has unique fixed point. Hence the operator equation v = M1vM2vM3v

has a solution.

Example 2: Given a closed and bounded interval $I = \left[\frac{1}{15}, \frac{15}{15}\right]$ in ℝ+ the set of real numbers, consider the nonlinear functional integral equation (in short FIE)

$$x(t) = [x(\alpha(t))]^2 \left[q(t) + \int_0^t g(u, x(\beta(u)))du\right]^2$$

(2.1)

for all $t \in I$, where $\alpha, \beta : I \rightarrow I$, $q : I \rightarrow$ ℝ+ and $g : I \times$ ℝ+ → ℝ+ are continuous.

By a solution of the FIE (1) we mean a continuous function *x*: *I* ⟶ ℝ+ that satisfies FIE (1) on *I*.

Let X = C(*I*, ℝ+) be a Banach algebra of all continuous real-valued function on *I* with the norm $||x|| = \sup_{t \in I} |x(t)|$. We shall obtain the solution of FIE (1) under some suitable conditions on the functions involved in (1).

Suppose that the function g satisfy the condition $|g(t,x)| \le 1 - q$, $||q|| < 1$ for all $t \in I$ and $x \in \mathbb{R}+$. Consider the three mapping M1, M2, M3 : X ⟶ X defined by

$$\mathrm{M}_1 x(t) = [x(\alpha(t))]^2,\ t \in I \text{ and } \mathrm{M}_2 x(t) = \left[q(t) + \int_0^t g(u, x(\beta(u)))du\right]^2,\ t \in I,$$

$$\mathrm{M}_2 x(t) = [x(\alpha(t))]^2,\ t \in I \text{ and } \mathrm{M}_3 x(t) = \left[q(t) + \int_0^t g(u, x(\beta(u)))du\right]^2,\ t \in I$$

$$\text{and } \mathrm{M}_3 x(t) = [x(\alpha(t))]^2,\ t \in I \text{ and } \mathrm{M}_1 x(t) = \left[q(t) + \int_0^t g(u, x(\beta(u)))du\right]^2,\ t \in I$$

Then the FIE (1) is equivalent to the operator equation $x(t)$ = M1$x(t)$ M2$x(t)$ M3$x(t)$, $t \in I$. Let H : X ⟶ X defined by H(y) = $\sqrt{y}$, $y \in$ X , where $\sqrt{y}(u) = \sqrt{y(u)}$,(positive square root) $u \in I$. Clearly, H is a homomorphism and it has a unique fixed point 1, where 1(t) = 1, $t \in I$. It is obvious that M1H = HM1, M2H = HM2 and M3H = HM3. So, 1 is a solution of FIE (1).

Theorem 3: Let A be a non-empty closed bounded and convex subset of a weakly compact Banach Algebra X. Let M1 : A⟶A , M2 : A⟶X and M3 : A⟶X be three mappings such that

(a) M1 satisfies asymptotically non-expansive mapping and $\lim_{n\to\infty}[\sup ||\mathrm{M}_1 x - \mathrm{M}_1^n x|| : x \in A] = 0$,

(b) M2, M3 are completely continuous and M = || M2 (A) M3(A) || < 1,

(c) I – M1 ⋄ M2 ⋄ M3 is demiclosed and M_1^nv M2w M3v' $\in A$ for v, w, v' ∈ A and n ∈ ℕ

then there exit a solution of the operator equation v = M1v M2v M3v(=(M1⋄M2⋄M3)v) in A.

Proof. First we show that I – M1 ⋄ M2 ⋄ M3 is closed. Let $c \in \overline{I - \mathrm{M1} \diamond \mathrm{M2} \diamond \mathrm{M3}}$ .Then there exist a sequence {cn} ⊆ I – M1 ⋄ M2 ⋄ M3 such that cn ⟶ c as $n \to \infty$. Since cn ∈ I – M1 ⋄ M2 ⋄ M3, so cn = (I – M1 ⋄ M2 ⋄ M3)zn for some zn ∈ X.

Since X is weakly compact so for every sequence {zn} in A there exist weakly convergence subsequence {zni} i.e. zni ⟶ z as $n \to \infty$. Now, zni – M1 ⋄ M2 ⋄ M3 zni ⟶ c as $n \to \infty$

Since I – M1 ⋄ M2 ⋄ M3 is demiclosed so c = (I – M1 ⋄ M2 ⋄ M3)z. Therefore c ∈ I – M1 ⋄ M2 ⋄ M3.

Hence I – M1 ⋄ M2 ⋄ M3 is closed.

For v, w, v' ∈ A, we define Fn : A⟶A by Fn (v) = $pn\,\mathrm{M}_1^n$v M2w M3v', where $pn = \frac{(1-\frac{1}{n})}{b_n}$ and {bn}⟶1 as $n \to \infty$.

Now, ||Fn (v) – Fn (p) – Fn (q)|| = || $pn\,\mathrm{M}_1^n$v M2w M3v'– $pn\,\mathrm{M}_1^n p$ M2w M3v'– $pn\,\mathrm{M}_1^n q$ M2w M3v'||

= pn || M2w M3v'|| || M_1^nv – $\mathrm{M}_1^n p$ – $\mathrm{M}_1^n q$||

≤ pn bn M||v – p – q|| = $\left(1 - \frac{1}{n}\right)$M||v – p – q|| ≤ M||v – p – q||

Since Fn is contraction and so it has unique fixed point Kn (v) ∈ A (say),

where Kn (v) = Fn (Kn (v)) = $pn\,\mathrm{M}_1^n$(Kn (v))M2v M3v.

Now, for any v, y, z ∈ A we have

||Kn (v) – Kn (y) – Kn (z)|| = || $pn\,\mathrm{M}_1^n$(Kn v)M2v M3v – $pn\,\mathrm{M}_1^n$(Kn y)M2y M3y – $pn\,\mathrm{M}_1^n$(Kn z)M2z M3z||

≤ pn || M_1^n(Kn v) – M_1^n(Kn y) – M_1^n(Kn z)|| ||M2v M3v|| + pn || M_1^n(Kn y) M3y|| ||M2v

– M2y – M2z|| + *pn* || M_1^n (*Kn* z) M2z|| ||M3v – M3y – M3z||

(2.2)
For fixed *a* ∈ *A*, we have

$$\|M_1^n(v)\| = \|M_1^n(v) - M_1^n(a) + M_1^n(a)\|$$
$$\leq bn\,\|v - a\| + \|M_1^n(a)\| = d \text{ (say)} < \infty$$

From equation (2.2),
we have

$$\|Kn(v) - Kn(y) - Kn(z)\| \leq \frac{d\,pn}{1 - M}\left(\|M2v - M2y - M2z\| + \|M3v - M3y - M3z\|\right)$$

So, *Kn* is completely continuous as M2, M3 are completely continuous. By Schauder's fixed point theorem *Kn* has a fixed point *xn* , say in *A*. Hence *xn* = *Kn xn* = *Fn* (*xn*) = *pn* M_1^n (*xn*) M2*xn* M3*xn*.

Now, *xn* – M_1^n *xn*M2*xn* M3*xn* = (*p*n – 1) M_1^n *xn*M2*xn* M3*xn* ⟶0 as *n* ⟶ ∞
(2.3)

||*xn* – M1*xn*M2*xn* M3*xn*|| ≤ ||*xn* – M_1^n *xn*M2*xn* M3*xn*|| + || M_1^n *xn*M2*xn* M3*xn* – M1*xn*M2*xn* M3*xn*||

= ||*xn* – M_1^n *xn*M2*xn* M3*xn*|| + ||M2*xn* M3*xn*|| || M_1^n *xn* – M1*xn*||
⟶ 0 as *n* ⟶ ∞ (by (2.3) and condition (a))

So, 0 ∈ *I* – M1 ⋄ M2 ⋄ M3 as *I* – M1 ⋄ M2 ⋄ M3 is closed.
Hence there exists a point *r*, such that 0 = (*I* – M1 ⋄ M2 ⋄ M3)*r*.
Hence the theorem follows.

Theorem 4: Let *B* be the bounded, open and convex subset with 0 ∈ *B* in a uniformly convex Banach algebra X. Let (M1, M2, M3) be three self mapping on $\overline{B}$ such that

(a) M1 satisfies uniformly *L*-Lipschitzian mapping on $\overline{B}$ and lim*n* ⟶∞$[\sup \| M_1 x - M_1^n x \| : x \in B] = 0$,

(b) M1 is demiclosed on $\overline{B}$ and *M* = ||M2(*B*)|| such that *LM* < 1

(c) M2, M3 are completely continuous and M_1^n v M2w + M3v ∈ *B* for v, w ∈ *B* and n ∈ ℕ

then there exit a solution of the operator equation v = M1v M2v + M3v(=(M1⋄M2)v + M3v)) in *B*.

Proof. Since M2 is a completely continuous, it is demicompact on $\overline{B}$. Also M1 is demicompact by (b). So for a sequence {*cn*} ∈ $\overline{B}$ such that $c_n - M1\,c_n \to a$, $c_n - M2\,c_n \to b$ as *n* ⟶∞ in $\overline{B}$, there exits subsequence $\{c_{n_k}\}$ such that $c_{n_k} \to c$ as $k \to \infty$, where c ∈ $\overline{B}$.

Since M1, M2 and M3 are continuous so $M_1 c_{n_k} \to$ M1c, $M_2 c_{n_k} \to$ M2c and $M_3 c_{n_k} \to$ M3c.

Now we show that *I* – M1 ⋄ M2 – M3 is closed.

Let z ∈ $\overline{I - M1 \diamond M2 - M3}$. Then for {zn} ⊆ (*I* – M1 ⋄ M2 – M3)cn such that zn ⟶ z as *n* ⟶∞.

We have as in **Theorem 3**,

$$c_{n_k} - M1 \diamond M2\, c_{n_k} - M3\, c_{n_k} \to z \text{ as } n \to \infty$$

Since *I* – M1 ⋄ M2 – M3 is continuous so c ∈ *I* – M1 ⋄ M2 – M3. Hence *I* – M1 ⋄ M2 – M3 is closed.

Define *Fn* : $\overline{B} \to \overline{B}$ by *Fn*(u) = *pn*(M_1^n uM2v + M3v), where {*pn*}⟶1 as *n* ⟶∞.

Now, ||*Fn* (u) – *Fn* (*p*)|| ≤ *p*n *LM* ||u – *p*||

Since *Fn* is contraction and so it has unique fixed point *Kn*v ∈ $\overline{B}$ (say)

Knv = $Fn(Kn$v) = $pn(M_1^n(Kn\text{v})$M2v + M3v). Now, for any v, y ∈ $\overline{B}$, we have

$||Kn(\text{v}) - Kn(\text{y})|| \leq pn\,||M_1^n(Kn\,\text{v}) - M_1^n(Kn\,\text{y})||\,||M2\text{v}|| + pn\,||M_1^n(Kn\,\text{y})||\,||M2\text{v} - M2\text{y}|| + ||M3\text{v} - M3\text{y}||$

(2.4)

For fixed $a \in \overline{B}$, we have

$$||M_1^n(u)|| \leq L||u - a|| + ||M_1^n(a)|| = d \text{ (say)} < \infty$$

From equation (4), we have

$$||Kn(\text{v}) - Kn(\text{y})|| \leq \frac{dp_n}{1-LM}||M2\text{v} - M2\text{y}|| + \frac{p_n}{1-LM}||M3\text{v} - M3\text{y}||$$

So, Kn is completely continuous as M2 and M3 are completely continuous. By Schauder's fixed point theorem Kn has a fixed point xn ,say in $\overline{B}$. Hence $xn = Kn\,xn = Fn(xn) = pn(M_1^n(xn)M2xn + M3(xn))$.

Now,

$$xn - M_1^n xn\ M2xn - M3xn = (pn - 1)\,(M_1^n xn\ M2xn + M3xn) \rightarrow 0 \text{ as } n \rightarrow \infty$$

(2.5)

$$||xn - M_1 xn\,M2xn - M3xn|| \leq ||xn - M_1^n xn\,M2xn - M3xn|| + ||M2\,xn||\,||M_1^n xn - M_1 xn||$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty \text{ (by (5) and condition (a))}$$

Since, 0 ∈ I – M1 ⋄ M2 – M3 and I – M1 ⋄ M2 – M3 is closed. Hence there exist a point r such that 0 = (I – M1 ⋄ M2 – M3)r. Hence the theorem follows.

If 0 ∉ B in the above **Theorem 4**.

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